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FLOATING BREAKWATERS IN SHALLOW WATER

by

J. J. Stoker, B. Fleishman, L. Weliczker [illegible]

**Prepared under Contract No. Nonr-285 (06)
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FLOATING BREAKWATERS IN SHALLOW WATER

by

J. J. Stoker, B. Fleishman, L. Weliczker [Wells]

This report represents results obtained at the Institute for Mathematics and Mechanics, New York University, under the auspices of Contract No. Nonr-285(06) with the Office of Naval Research.

The object of this report is to derive the general hydrodynamical theory needed for mathematical treatment of problems involving the interaction between bodies immersed in water and surface gravity waves in the water. In particular, the problem of the effectiveness of floating breakwaters in the form of rigid bodies or of beams with bending flexibility will be treated; in these cases the principal object is to determine the amount of reflection of an incoming progressing wave that can be achieved by such structures in terms of parameters which characterize the wave motion and the physical nature of the floating breakwater.

The theory presented here is an approximate theory derived from the exact hydrodynamical theory for gravity waves of small surface amplitude by making the simplifying assumption that the wave length λ in the water is sufficiently large compared with the depth h of the water. If the ratio λ/h is of the order eight to ten the results should be reasonably accurate, and such a condition would often be realized in cases having practical interest. It is entirely possible to formulate a mathematical theory without making such a restrictive assumption regarding the ratio of wave length to depth -- for floating rigid bodies having simple harmonic motions, for example, the theory has been worked out by F. John* -- but the difficulties involved in obtaining solutions in cases of interest in practice are so formidable that no progress in this direction has been achieved.

The material is presented here in three parts. The first section (by J. J. Stoker) consists of a derivation of a theory of floating bodies in shallow water which is carried out in such a way as to permit floating bodies of widely different physical characteristics: rigid bodies, inert distributions of mass, membranes under tension, or beams with bending stiffness, for example. In Section 2 (by B. Fleishman), the details of

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* On the Motion of Floating Bodies, I. Comm. on Pure and Appl. Math., Vol. II, 1949, p. 13.

the theory are worked out for floating rigid bodies and numerical results are given for a floating board having one degree of freedom (a purely vertical motion), and two degrees of freedom (angular and vertical motion). Reflection and transmission coefficients are calculated as well as the motion of the board and the dynamic pressure on it. The results are shown in the form of curves. In Section 3 (by L. Weliczker) the relevant theory is derived and applied to floating beams of infinite, semi-infinite, and finite length. (It might be mentioned that this was the most difficult part of the investigation.)

The case of a floating beam was suggested to us by J. H. Carr of the Hydraulics Structures Laboratory at the California Institute of Technology as one having practical possibilities; at his suggestion calculations in specific numerical cases have been carried out in order to determine the effectiveness of such a breakwater. The reason for considering such a structure for a breakwater as a means of creating relatively calm water between it and the shore is the following: a structure which floats on the surface without sinking far into the water need not be subjected to large horizontal forces and hence would not necessarily require a massive anchorage. However, in order to be effective as a reflector of waves such a floating structure would probably have to be built with a fairly large dimension in the direction of travel of the incoming waves. As a consequence of the length of the structure, it would be bent like a beam under the action of the waves and hence could not in general be treated with accuracy as a rigid body in determining its effectiveness as a barrier. This brings with it the possibility that the structure might be bent sufficiently that the stresses set up would be a limiting feature in the design. The main object of the present report is to present a means of studying all of these questions theoretically and to make calculations in a number of specific cases.

The floating beam is assumed to be placed at right angles to the direction of the incoming waves (i.e. its long axis is at right angles to the wave crests). The water is supposed to

extend to infinity in both of the directions along the axis of the beam. The specifications (as suggested by Carr) for a beam having a width of one foot (parallel to the wave crests, that is) were:

Weight: 85 pounds/foot

Moment of inertia (of area of cross-section): $0.2(\text{ft.})^4$

Modulus of elasticity: $30 \times 10^6(\text{pounds/in.}^2)$ or
$437 \times 10^7(\text{pounds/ft.})^2$.

The depth of the water is taken as 40 feet. Simple harmonic progressing waves having periods of 8 and of 15 secs. were to be considered, and these correspond to wave lengths of 250 and 507 feet, and to circular frequencies σ of 418×10^{-3} and 785×10^{-3} cycles per second, respectively. The problem is to determine the reflecting power of the beam under these circumstances when the length of the beam is varied. In other words, we assume a wave train to come from, say, the right hand side of the beam and that it is partly transmitted under the beam to the left hand side and partly reflected back to the right hand side. The ratio R/B of the amplitude R of the reflected wave and the amplitude B of the incoming wave is a measure of the effectiveness of the beam as a breakwater. The effectiveness of the breakwater can also be measured in terms of the transmitted and reflected energies. If T is the amplitude of the transmitted wave the energy balance is expressed by the relation $(T/B)^2 + (R/B)^2 = 1$, so that $(R/B)^2$, for example, represents the fraction of the incoming energy which is reflected.

Before discussing the case of beams of finite length it is interesting and worthwhile to consider semi-infinite beams first. Since the calculations are easier than for beams of finite length it was found possible to consider a larger range of values of the parameters than was given above. The results are summarized in the following tables:

Table A

λ (ft)	$\sigma\ (\frac{1}{\sec})$	W(pounds)	I $(ft)^4$	E $\left(\frac{\text{pounds}}{ft^2}\right)$	h (ft)	R/B
507	.418	85	.20	437×10^7	40	.14
250	.785	85	.20	437×10^7	40	.19
196	1.0	85	.20	437×10^7	40	.23
147	1.5	85	.20	437×10^7	40	.32
98	2.0	85	.20	437×10^7	40	.43

In this table the beam design data are as given above. At the two specified circular frequencies of 0.418 and 0.785 one sees that the floating beam is quite ineffective as a breakwater since the reflected wave has an amplitude of less than 1/5 of the amplitude of the incoming wave, even for the higher frequency (and hence shorter wave length), which means that less than 4% of the incoming energy is reflected back. At higher frequencies, and hence smaller wave lengths, the breakwater is more effective, as one would expect. However, the approximate theory used to calculate the reflection coefficient R/B can be expected to be accurate only if the ratio λ/h of wave length to depth is sufficiently large, and even for the case $\lambda = 250$ ft. ($\sigma = .785$) the reflection coefficient of value 0.19 may be in error by perhaps 10% or more since λ/h is only about 6, and the errors for the shorter wave lengths would be greater.

Table B

λ	σ	W	I	E	h	R/B
507	.418	384	.20	437×10^7	40	.51
250	.785	384	.20	437×10^7	40	.75

The only change as compared with the first two rows of table A is that the weight per foot of the beam has been increased by a

factor of more than 4 from a value of 85 pounds/ft to a value of 384 pounds/ft. The result is a decided increase in the effectiveness of the breakwater, especially at the shorter wave length, since more than half (i.e. $(.75)^2$) of the incoming energy would be reflected back. However, this beneficial effect is coupled with a decided disadvantage, since quadrupling the weight of the beam would cause it to sink deeper in the water in like proportion and hence might make heavy anchorages necessary.

Table C

λ	σ	W	I	E	h	R/B
507	.418	85	2.0	437×10^7	40	.26
250	.785	85	2.0	437×10^7	40	.32
			∞			1

Table C is the same as the first two rows of table A except that the bending stiffness has been increased by a factor of 10 by increasing the moment of inertia of the beam cross-section from 0.2 ft^4 to 2.0 ft^4. Such an increase in stiffness results in a noticeable increase in the effectiveness of the breakwater, but by far not as great an increase as is achieved by multiplying the weight by a factor of four. If the stiffness were to be made infinite (i.e. if the beam were made rigid) the reflection coefficient could be made unity, and no wave motion would be transmitted. This is evidently true for a semi-infinite beam, but would not be true for a rigid body of finite length.

Table D

λ	σ	W	I	E	h	R/B
507	.418	85	0	0	40	.001
250	.785	85	0	0	40	.007

Here, the difference as compared with table A is that the beam stiffness is taken to be zero. This means that the surface of the water is assumed to be covered by a distribution of inert particles weighing 85 pounds per foot. As we observe, there is practically no reflection and this is not surprising since the mass distribution has a value that is only slightly greater than that of water itself (62.4 pounds per foot).

One might perhaps summarize the above results as follows: A very long beam can be effective as a floating breakwater if it is stiff enough. However, a reasonable value for the stiffness (the value 0.2 given above) leads to an ineffective breakwater unless the specific weight of the beam is a fairly large multiple (say 8 or 10) of the weight of water.

In practice it seems unlikely that beams long enough to be considered semi-infinite would be practicable as breakwaters. (The term "long enough" might be interpreted to mean a sufficiently large multiple of the wave length, but since the wave lengths are of the order of 200 feet or more the correctness of this statement seems obvious.) It therefore is necessary to investigate the effectiveness of beams of finite length. Such an investigation requires extremely tedious calculations -- so much so that only a certain number of numerical cases have been treated. These are summarized in the following tables.

$\sigma = .785$, $\lambda = 250$

(ft)	R/B
17.5	0
49.2	.93
72.9	0
98.5	.75
145.9	.10
196.9	0
291.8	.33
450.4	.32
583.6	.12
656.3	.13
875.4	.32
∞	.19

$\sigma = .418$, $\lambda = 507$

(ft)	R/B
145.9	.17
196.9	.53
291.8	.13
443.0	.90
583.6	.74
656.2	.62
874.9	.07
875.4	.08
940.3	.54
∞	.14

In these tables the parameters have values the same as in the first two rows of table A, except that now lengths other than infinite length are considered. The most noticeable feature of the results given in the tables is their irregularity and the fact that at certain lengths--even certain rather short lengths--the beam proposed by Carr seems to be quite effective. For example, when the wave length is 250 ft. a beam less than 50 ft. long reflects more than 80% of the incoming energy. A beam of length 443 ft. is also equally effective at the longer wave length of 507 ft.*

However, the maximum effectiveness of any such breakwater occurs for a specific wave length within a certain range of wave lengths; thus the reflection of a given percentage of the incoming wave energy would involve changing the length (or some other parameter) of the structure in accordance with changes in the wave length of the incoming waves. Also, the reflection coefficient seems to be rather sensitive to changes in the parameters, particularly for the shorter structures: a relatively slight change in length from an optimum value (or a slight change in frequency) leads to a sharp decrease in the reflection coefficient. It is also possible--even probable--that the shallow water approximation used here as a basis for the theory is not sufficiently accurate for a floating beam whose length is too much less than the wave length, and we are now trying to learn something about the magnitude of the errors committed in using the shallow water theory in such cases. Nevertheless, it does seem possible to design floating breakwaters of reasonable length which would be effective at a given wave length. Perhaps it is not too far-fetched to imagine that

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*It might not be amiss to consider the physical reason why it is possible that a beam of finite length could be more effective as a breakwater than a beam of infinite length. Such a phenomenon comes about, of course, through multiple reflections that take place at the ends of the beam. Apparently the phases sometimes arrange themselves in the course of these complicated interactions in such a way as to yield a small amplitude for the transmitted wave. That such a process might well be sensitive to small changes in the parameters, as is noted in the discussion, cannot be wondered at.

sections could be added to or taken away from the breakwater in accordance with changing conditions.

Another consequence of the theory--which is also obvious on general grounds--is that there is always the chance of creating a large standing wave between the shore and the breakwater because of reflection from the shore, unless the waves break at the shore; this effect is perhaps not important if the main interest is in breakwaters off beaches of not too large slope since breaking at the shore line then always occurs. (The theory developed here could be extended to cases in which the shore reflects all of the incoming energy, it might be noted.) In principle, the calculation of the deflection curve of the structure, and hence also of the bending stresses in it, as given by the theory is straightforward, but it is very tedious; consequently only the reflection coefficients have been calculated.

1. Theory for Floating Obstacles in Shallow Water.

By J. J. Stoker

We suppose the water to fill a region lying above a fixed horizontal surface (the bottom) $y = -h$, and beneath a surface $y = Y(x,z;t)$, the motion of which is for the time being supposed known (cf. fig. 1.1). The y-axis is taken vertically upward, and the x,z-plane is horizontal. The upper surface of the water given by $y = Y(x,z;t)$ will consist partly of the free surface (to be determined by the condition that the pressure vanishes there) and partly of the surface of immersed bodies; it is, however, not necessary to specify more about this surface for the present except that it should represent always a small displacement from a rest position of equilibrium of the combined system consisting of water and immersed bodies.

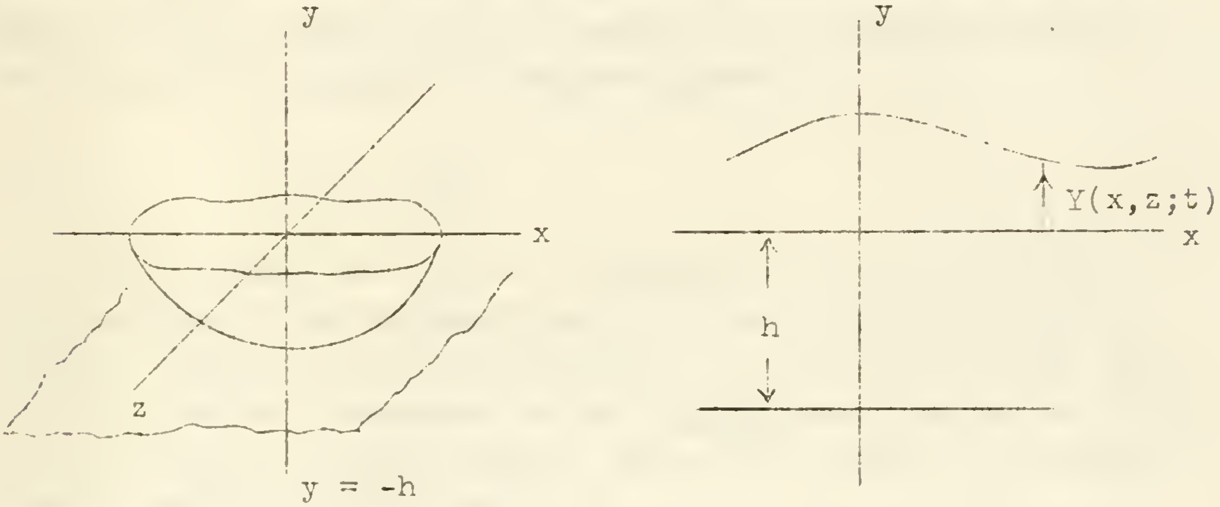

fig. 1.1

We assume that the motion is irrotational and hence that a velocity potential $\underline{\Phi}(x,y,z;t)$ exists such that the velocity $\vec{q}$ of the water particles is given by $\vec{q} = \operatorname{grad} \underline{\Phi}$; this function satisfies the Laplace equation

$$(1.1) \qquad \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

in the space filled by the water. The letter subscripts here, and in what follows, denote differentiations. Since we deal with small motions from the state of rest, we may linearize the basic hydrodynamical theory and ignore all quadratic and higher order terms in Φ and its derivatives. In addition, it is then legitimate to assume that all boundary conditions at the upper surface of the water are to be satisfied at the equilibrium position; this position is supposed given by

$$(1.2) \qquad y = \bar{\eta}\,(x,z) \quad .$$

In effect, we are dealing with problems which concern small oscillations in the neighborhood of a rest position of equilibrium. (The bar over the quantity η points to the fact that $\bar{\eta}$ could also be interpreted as the average position of the water in the important special case in which the motion is a simple harmonic motion in the time.)

At points on the free surface, $\bar{\eta}$ will of course be zero, and this in turn means in view of (1.2) that the x,z-plane is taken in the undisturbed position of the free surface. Under the immersed bodies the value of $\bar{\eta}$ will be fixed by the static equilibrium position of the given bodies. Thus $\bar{\eta}$ is in all cases a given function of x and z; for a floating rigid body, for example, it would be determined by Archimedes' law.

The condition to be satisfied at the upper surface is the kinematic condition:

$$(1.3) \qquad \Phi_x Y_x + \Phi_z Y_z - \Phi_y + Y_t = 0 \quad ,$$

which states that a particle once on the surface remains on it. At the bottom surface, of course, the condition to be satisfied is

$$(1.3)_1 \qquad \Phi_y = 0 \quad ,$$

as one sees on physical grounds as well as from (1.3) applied to the case Y = constant. In addition, Bernoulli's law for determining the pressure at the upper surface takes the form

$$(1.4) \qquad \frac{p}{\rho} + \Phi_t + gy = 0 \quad \text{for} \quad y = Y$$

when quadratic terms in the velocity components are neglected. We now write the equation of the moving upper surface in the form

$$(1.5) \qquad y = Y(x,z;t) = \bar{\eta}(x,z) + \eta(x,z;t)$$

and assume in accordance with out statement above that $\eta(x,z;t)$ represents a small vertical displacement from the equilibrium position given by $y = \bar{\eta}$. Upon insertion in (1.3) and (1.4) we find upon ignoring quadratic terms in η and Φ and their derivatives:

$$\left.\begin{aligned} &(1.6) \qquad \Phi_x \bar{\eta}_x + \Phi_z \bar{\eta}_z - \Phi_y + \eta_t = 0 \\ &(1.7) \qquad \frac{p}{\rho} + \Phi_t + g(\bar{\eta} + \eta) = 0 \end{aligned}\right\} \quad \text{at} \quad y = \bar{\eta}(x,z)$$

as boundary conditions to be satisfied at $y = \bar{\eta}(x,z)$ in accordance with remarks made above.

At points corresponding to a free surface where $p = 0$ we would have, for example, $\bar{\eta} = 0$ and hence

$$\left.\begin{aligned} &(1.8) \qquad -\Phi_y + \eta_t = 0 \\ &(1.9) \qquad \Phi_t + g\eta = 0 \end{aligned}\right\} \quad \text{at} \quad y = 0 \quad .$$

These are the usual free surface conditions (cf. Lamb, p.364). Another special case might be that in which the motion of a portion of the upper surface is prescribed, i.e. $\eta(x,z;t)$ as well as $\bar{\eta}$ would be presumed known; in such a case the condition

(1.6) alone would suffice as a boundary condition for the harmonic function Φ. In the problems to be treated here, however, we do not wish to assume that the motion of the immersed body is known in advance; rather, it is to be determined by the interaction with the water which exerts a pressure $p(x,z;t)$ on it in accordance with (1.7). Thus the exact formulation of our problems would require the determination of a harmonic function $\Phi(x,y,z;t)$ in the space between $y = -h$ and $y = \bar{\eta}$ which satisfies the conditions (1.6) and (1.7) at the upper surface (in particular the conditions (1.8) and (1.9) on the free surface) and $(1.3)_1$ at the bottom. Additional conditions where immersed bodies occur (that must be obtained from the appropriate dynamical conditions for such bodies) would be necessary to determine the pressure p, which provides the "coupling" between the water on the one hand and the immersed bodies on the other. Finally, appropriate initial conditions for the water and the immersed bodies at the initial instant would be needed if one were to study non-steady motions, or--as will be the case here--conditions at ∞ of the radiation type are needed if simple harmonic motions (that is, steady vibrations instead of transients) are studied. It need hardly be said that the difficulties of carrying out the solutions of such problems are very great indeed--so much so that we turn to an approximate theory which is based on the assumption that the depth of the water is sufficiently small and that the immersed bodies are rather flat.*

In the derivation of the approximate theory we start from the potential equation (1.1) for Φ and integrate it with respect to y from the bottom to the equilibrium position** of the top surface $y = \bar{\eta}(x,z)$ to obtain:

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* In the course of the derivation the terms neglected are given explicitly so that a precise statement about them can be made.

** One sees readily that carrying out the integrations to $y = \bar{\eta}$ rather than to $y = \bar{\eta} + \eta$ yields the same results within terms of second order.

$$(1.10)\qquad \int_{-h}^{\bar{\eta}} \Phi_{yy}\, dy = -\int_{-h}^{\bar{\eta}} (\Phi_{xx} + \Phi_{zz})\, dy = \bar{\Phi}_y$$

$$= -\frac{\partial}{\partial x}\int_{-h}^{\bar{\eta}} \Phi_x\, dy - \frac{\partial}{\partial z}\int_{-h}^{\bar{\eta}} \Phi_z\, dy + \bar{\eta}_x\, \bar{\Phi}_x + \bar{\eta}_z\, \bar{\Phi}_z \quad .$$

Here, and in what follows, a bar over the quantity Φ means that it is to be evaluated at the equilibrium position of the upper surface of the water, i.e. for $y = \bar{\eta}(x,z)$. From the kinematic surface condition (1.6) we have therefore (due regard being paid to the fact that a bar should now be put over Φ in (1.6)):

$$(1.11)\qquad \eta_t = -\frac{\partial}{\partial x}\int_{-h}^{\bar{\eta}} \Phi_x\, dy - \frac{\partial}{\partial z}\int_{-h}^{\bar{\eta}} \Phi_z\, dy \quad .$$

This condition--really a continuity condition--expresses the fact that the water is incompressible. Consider next the result of integrating by parts the right hand side of (1.11); in particular:

$$(1.12)\qquad \int_{-h}^{\bar{\eta}} \Phi_x\, dy = \bar{\eta}\, \bar{\Phi}_x + h\, \underline{\Phi}_x - \int_{-h}^{\bar{\eta}} y\, \Phi_{xy}\, dy$$

in which the bar under a symbol means that its value is taken for $y = -h$. Since we have

$$(1.13)\qquad \int_{-h}^{\bar{\eta}} h\, \Phi_{xy}\, dy = h\, \bar{\Phi}_x - h\, \underline{\Phi}_x$$

we may eliminate $\underline{\Phi}_x$ from (1.12) to obtain:

$$(1.14)\qquad \int_{-h}^{\bar{\eta}} \Phi_x\, dy = (\bar{\eta} + h)\, \bar{\Phi}_x - \int_{-h}^{\bar{\eta}} (h + y)\, \Phi_{xy}\, dy \quad .$$

Indeed, we have quite generally for any function $F(x,y,z,t)$ the result:

$$(1.14)_1\qquad \int_{-h}^{\bar{\eta}} F\, dy = (\bar{\eta} + h)\, \bar{F} - \int_{-h}^{\bar{\eta}} (h + y)\, F_y\, dy \quad .$$

Making use of the analogous expression for the integral of Φ_z we obtain from (1.11) the relation

$$(1.15) \qquad \eta_t = -[(\bar{\eta} + h)\,\Phi_x]_x - [(\bar{\eta} + h)\,\Phi_z]_z + I_x + J_z$$

in which

$$(1.16) \qquad I = \int_{-h}^{\bar{\eta}} (h + y)\,\Phi_{xy}\,dy, \qquad J = \int_{-h}^{\bar{\eta}} (h + y)\,\Phi_{zy}\,dy.$$

In addition, we have from (1.10) in combination with (1.14) the condition:

$$(1.17) \qquad \bar{\Phi}_y = -(\bar{\eta} + h)[\bar{\Phi}_{xx} + \bar{\Phi}_{zz}] + I_x + J_z,$$

as one can readily verify.

Up to this point we have made no approximations other than those arising from linearizing for small oscillations. The essential step in obtaining our approximate theory is now taken in neglecting the terms I_x and J_z. This in turn is justified if it is assumed that certain second and third derivatives of Φ are bounded when h is small and that $\bar{\eta}$ and its first derivatives are small of the same order as h: one sees that the terms I_x and J_z in the right hand sides of (1.15) and (1.17) are then of order h^2 while the remaining terms are of order h. Under the free surface in the case of a simple harmonic oscillation one can show that this approximation requires the depth to be small in comparison with the wave length, as we shall see in a later section.

Upon differentiating the relation (1.7) for the pressure at the upper surface of the water with respect to t (again noting that a bar should be placed over the term Φ_t in (1.7)) and using (1.15), we find the equation

$$(1.18) \qquad \frac{1}{g}\,\bar{\Phi}_{tt} + \frac{p_t}{g\rho} = [(\bar{\eta} + h)\,\bar{\Phi}_x]_x + [(\bar{\eta} + h)\,\bar{\Phi}_z]_z$$

after dropping the terms I_x and J_z. This is the basic

differential equation for the function $\bar{\Phi}(x,z)$ which holds everywhere on the upper surface of the water. In particular, we have at the free surface where $p = 0$ and $\bar{\eta} = 0$ the equation

$$\bar{\Phi}_{xx} + \bar{\Phi}_{zz} - \frac{1}{gh}\bar{\Phi}_{tt} = 0, \tag{1.19}$$

that is, the linear wave equation in the two space variables x,z and the time t. As a consequence, all disturbances propagate with the constant speed $c = \sqrt{gh}$, as is well known for this equation.

If there is an immersed object in the water, the equation (1.19) holds everywhere in the x,z-plane exterior to the curve C which defines the water line on the immersed body in its equilibrium position. The curve C is supposed given by the equations

$$x = x(s), \quad z = z(s) \tag{1.20}$$

in terms of a parameter s. We must have boundary, or perhaps it is better to say, transition conditions at the curve C which connect the solutions of (1.19) in the exterior of C in an appropriate manner with the motion of the water under the immersed body. Reasonable conditions for this purpose can be obtained from the laws of conservation of mass and energy. Consider an element of length ds of the curve C representing the water line of the immersed body (cf. fig. 1.2). The

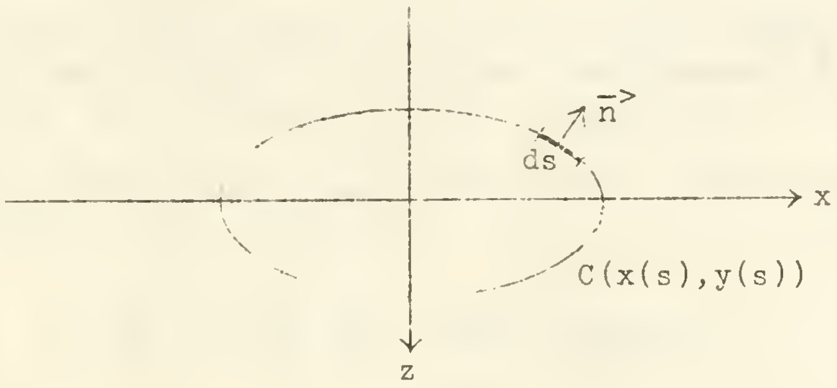

fig. 1.2

expression

$$\left\{ \rho \int_{-h}^{\bar{\eta}} \Phi_n \, dy \right\} ds$$

represents the mass flux through a vertical strip having normal n and extending from the bottom to the top of the water. From $(1.14)_1$ applied for $F = \Phi_n$ we have

$$(1.21) \quad \rho \int_{-h}^{\bar{\eta}} \Phi_n \, dy = \rho(\bar{\eta} + h)\, \bar{\Phi}_n - \rho \int_{-h}^{\bar{\eta}} (h + y)\, \Phi_{ny} \, dy$$

$$\simeq \rho(\bar{\eta} + h)\, \bar{\Phi}_n$$

where the second term is ignored because it is of order h^2. Thus it would be reasonable to require that $(\bar{\eta} + h)\, \bar{\Phi}_n$ should be continuous on C since this is the same as requiring that the mass of the water is conserved within terms of the order retained otherwise in our theory. For the flux of energy across a vertical strip with normal $\vec{n}$ we have

$$(1.22) \quad \left\{ \int_{-h}^{\bar{\eta}} p \, \Phi_n \, dy \right\} ds = \left\{ -\rho \int_{-h}^{\bar{\eta}} \Phi_t \, \Phi_n \, dy - g\rho \int_{-h}^{\bar{\eta}} y \, \Phi_n \, dy \right\} ds$$

upon making use of the Bernoulli law (1.4) for the pressure p. Once more we may ignore the second term in the brackets since it is of order h^2. Upon applying $(1.14)_1$ with $F = \Phi_t \, \Phi_n$ and again ignoring a term of order h^2 we find

$$(1.23) \quad \left\{ \int_{-h}^{\bar{\eta}} p \, \Phi_n \, dy \right\} ds = \left\{ -\rho(\bar{\eta} + h)\, \bar{\Phi}_n \, \bar{\Phi}_t \right\} ds.$$

Since we have already required that $(\bar{\eta} + h)\, \bar{\Phi}_n$ should be continuous, we see that the additional requirement, $\bar{\Phi}_t$ continuous, ensures the continuity of the energy flux.

As transition conditions on the curve C delimiting the immersed body we have therefore

$$(1.24) \qquad (\bar{\eta} + h)\, \bar{\Phi}_n, \ \bar{\Phi}_t \quad \text{continuous on C.}$$

Of course, if $\bar{\eta}$ is continuous (e.g. if the sides of the body in contact with the water do not extend vertically below the undisturbed free surface) it follows that $\overline{\overline{\Phi}}_n$ would then be continuous.

In order to make further progress it is necessary to specify the properties of the immersed body. However, we have succeeded in obtaining the equation (1.18) which is generally valid and of basic importance for our theory together with the transition conditions (1.24) valid at the edge of immersed bodies; in particular, we have the definitive equation for the free surface itself in the form of the linear wave equation (1.19). The idea behind this method of approximation is to get rid of the depth variable by an integration over the depth, so that the problems then are considered only in the x,z-plane. At the same time, the problems are no longer problems in potential theory in three space variables, but rather problems involving the wave equation with only two space variables, and hence they are more open to attack by known methods. How this comes about will be seen in special cases in the next two sections.

2. The Effect of a Floating Rigid Body on Simple Harmonic Waves.*

By B. Fleishman

We consider only two-dimensional motions of a rigid body in the form of a thin plane slab (cf. fig. 2.1). The ends of the slab are at $x = \pm a$. In accordance with the theory presented above we must determine the surface value $\underline{\Phi}(x,t)$ and the displacement $\eta(x,t)$ of the board from the differential equations (cf. (1.15), (1.19)):

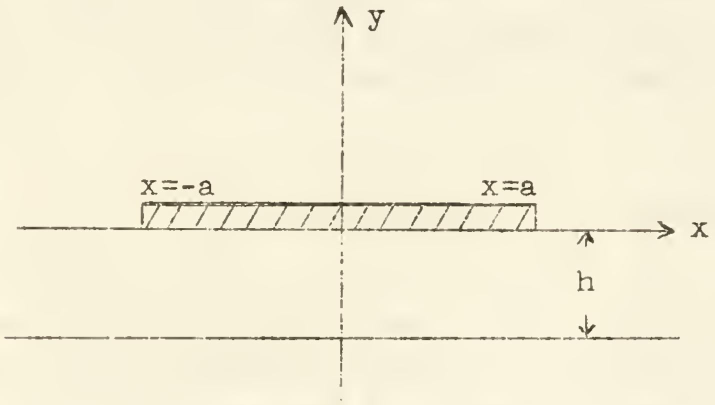

fig. 2.1

$$\underline{\Phi}_{xx} = \frac{1}{gh}\underline{\Phi}_{tt}, \quad |x| > a \tag{2.1}$$

$$\eta_t = -h\,\underline{\Phi}_{xx}, \quad |x| < a. \tag{2.2}$$

We have dropped the bar over the quantity $\underline{\Phi}$. We have also assumed that $\bar{\eta}(x)$ for $|x| < a$, the rest position of equilibrium of the board, is zero; this is an approximation that is justified because we assume that the board is so light that it does not sink appreciably below the water surface when in equilibrium. (This assumption is by no means necessary--it would not

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[*] These cases have been treated by F. John in the paper cited in the introduction.

be difficult to deal with the problem if this simplifying assumption were not made.)

Since $\overline{\eta}$ is zero, it follows (cf. (1.24)) that the transition conditions at the ends of the board are

$$\Phi_x, \Phi_t \quad \text{continuous at} \quad x = \pm a. \tag{2.3}$$

We are interested in the problem of the effectiveness of the floating board as a barrier to a train of waves coming from the right ($x = +\infty$). The equation (2.1) has as its general solution

$$\Phi(x,t) = F(x - ct) + G(x + ct), \qquad c = \sqrt{gh}$$

in terms of two arbitrary functions F and G (as one can readily verify) which clearly represent a superposition of two progressing waves moving to the right and to the left, respectively, with the speed $\sqrt{gh}$. It is natural, in our present problem, to expect that for $x > a$ there would exist in general both an incoming and an outgoing wave because of reflection from the barrier, while for $x < a$ we would prescribe only a wave going outward (i.e. to the left). We shall see that these qualitative requirements lead to a unique solution of our problem.

We consider only simple harmonic waves; it is thus natural to write

$$\Phi(x,t) = \phi(x)\, e^{i\sigma t}, \qquad |x| > a, \tag{2.4}$$

$$\eta(x,t) = v(x)\, e^{i\sigma t}, \qquad -a < x < a, \tag{2.5}$$

with the stipulation that the real part is to be taken at the end. (It is necessary also to permit $\phi(x)$ and $v(x)$ to be complex-valued functions of the real variable x.) The conditions (2.1) and (2.2) now become

(2.6) $$\frac{d^2\phi}{dx^2} + \frac{\sigma^2}{gh}\phi = 0, \qquad |x| > a$$

(2.7) $$\frac{d^2\phi}{dx^2} + \frac{i\sigma}{h} v = 0, \qquad |x| < a.$$

The equation (2.6) has as general solution

(2.8) $$\phi(x) = Ae^{ikx} + Be^{-ikx}$$

with k given by

(2.9) $$k = \sigma/\sqrt{gh} .$$

For $\Phi(x,t)$ we have therefore

(2.10) $$\Phi(x,t) = Ae^{i(kx-\sigma t)} + Be^{i(kx+\sigma t)},$$

the first term representing a progressing wave moving to the right, the second a wave moving to the left. In our problems we prescribe the incoming wave from the right, and hence for $\phi(x)$ we write

(2.11) $$\phi(x) = Be^{ikx} + Re^{-ikx}, \qquad x > a,$$

in which B is prescribed, while R--the amplitude of the reflected wave (more precisely, $|R|$ is its amplitude)--is to be determined. At the left we write

(2.12) $$\phi(x) = Te^{ikx},$$

with T--the amplitude of the transmitted wave--to be determined.

To complete the formulation of the problem it is necessary to consider the dynamics of our floating rigid body. We shall treat two cases: a) the board is held rigidly fixed in a horizontal position, b) the board floats freely in the water.

a) Rigidly Fixed Board.

If the board is rigidly fixed we have $\eta(x,t) \equiv 0$, and hence (cf. (2.5)) $v(x) \equiv 0$. It follows from (2.7) that ϕ_{xx} vanishes identically under the board and hence that $\phi(x)$ is a linear function:

$$\phi(x) = \gamma x + \delta, \quad -a < x < +a. \tag{2.13}$$

Since $\Phi_x(x,t)$ furnishes the horizontal velocity component of the water, it follows from (2.13) and (2.4) that the velocity under the board is given by $\gamma e^{i\sigma t}$, i.e. it is constant everywhere under the board at each instant--a not unexpected result.

We now write down the transition conditions at $x = \pm a$ from (2.3), making use of (2.4) and of (2.11) at $x = +a$ and (2.12) at $x = -a$; the result is:

$$\left\{\begin{aligned} Be^{ika} + Re^{-ika} &= \gamma a + \delta \\ Be^{ika} - Re^{-ika} &= \gamma/ik \\ Te^{-ika} &= -\gamma a + \delta \\ Te^{-ika} &= \gamma/ik. \end{aligned}\right. \tag{2.14}$$

Once the real number B--the amplitude of the incoming wave--has been prescribed, these four equations serve to fix the constants R, T, γ, and δ and hence the functions $\Phi(x,t)$ and $\eta(x,t)$. The pressure under the board can then be determined (cf. the expression (1.4) for Bernoulli's law) from

$$p(x,t) = -\rho\, \Phi_t = -\rho i\sigma\, \phi(x) e^{i\sigma t}. \tag{2.15}$$

(Observe that the quantity Y in (1.4) is zero in the present case.)

In terms of the dimensionless parameter

$$\theta = 2a/\lambda\ , \tag{2.16}$$

the ratio of the length of the board to the wave length λ (cf. (2.10)) on the free surface given by

$$\lambda = 2\pi/k, \tag{2.17}$$

the solution of (2.14) is

$$\begin{aligned} R &= \frac{\theta\pi i B e^{2\theta\pi i}}{\theta\pi i + 1} \\ T &= \frac{B e^{2\theta\pi i}}{\theta\pi i + 1} \\ \nu &= \frac{\theta\pi i}{a} \cdot \frac{B e^{\theta\pi i}}{\theta\pi i + 1} \\ \delta &= B e^{\theta\pi i}. \end{aligned} \tag{2.18}$$

The reflection and transmission coefficients are obtained at once:

$$\begin{aligned} C_r &= \frac{|R|}{|B|} = \frac{\theta\pi}{\sqrt{1 + \theta^2\pi^2}}, \\ C_t &= \frac{|T|}{|B|} = \frac{1}{\sqrt{1 + \theta^2\pi^2}}. \end{aligned} \tag{2.19}$$

They depend only upon the ratio $\theta = 2a/\lambda$, as one would expect. They also satisfy the relation $C_r^2 + C_t^2 = 1$, as they should: this is an expression of the fact that the incoming and outgoing energies are the same. The following table gives a few specific values for these coefficients:

θ	C_t	C_r
0.5	.54	.85
1.0	.30	.95
2.0	.157	.986

Thus a fixed board whose length is half the incoming wave length has the effect of reducing the amplitude behind it by about 50 percent and of reflecting about 72 percent of the incoming energy.

It is of interest to study the pressure variation under the board. This is given in the present case (cf. (2.15)) by

$$p(x,t) = -i\sigma\rho(\nu x + \delta)e^{i\sigma t}, \tag{2.20}$$

the real part only to be taken. Thus the pressure varies linearly in x, but it is a different linear function at different times since ν and δ are complex constants and the real part only is to be taken. The result of taking the real part of the right hand side of (2.20) can be readily put in the form

$$p(x,t) = p_1(x)\cos\sigma t - p_2(x)\sin\sigma t \tag{2.21}$$

with

$$\left.\begin{aligned} p_1(x) &= \sigma\rho B(b_1(x)\sin r + b_2(x)\cos r) \\ p_2(x) &= \sigma\rho B(b_2(x)\sin r - b_1(x)\cos r) \end{aligned}\right\}, \quad r = \theta\pi \tag{2.22}$$

and

$$\begin{aligned} b_1(x) &= \frac{r^2}{1+r^2}\frac{x}{a} + 1 \\ b_2(x) &= \frac{r}{1+r^2}\frac{x}{a}\,. \end{aligned} \tag{2.23}$$

We have assumed in making these calculations, as stated above, that B, which represents the amplitude of the incoming wave, is a real number.

In fig. 2.2 the results of computations for the pressure distribution for time intervals of 1/4 cycle over the full period are given for a special numerical case in which the

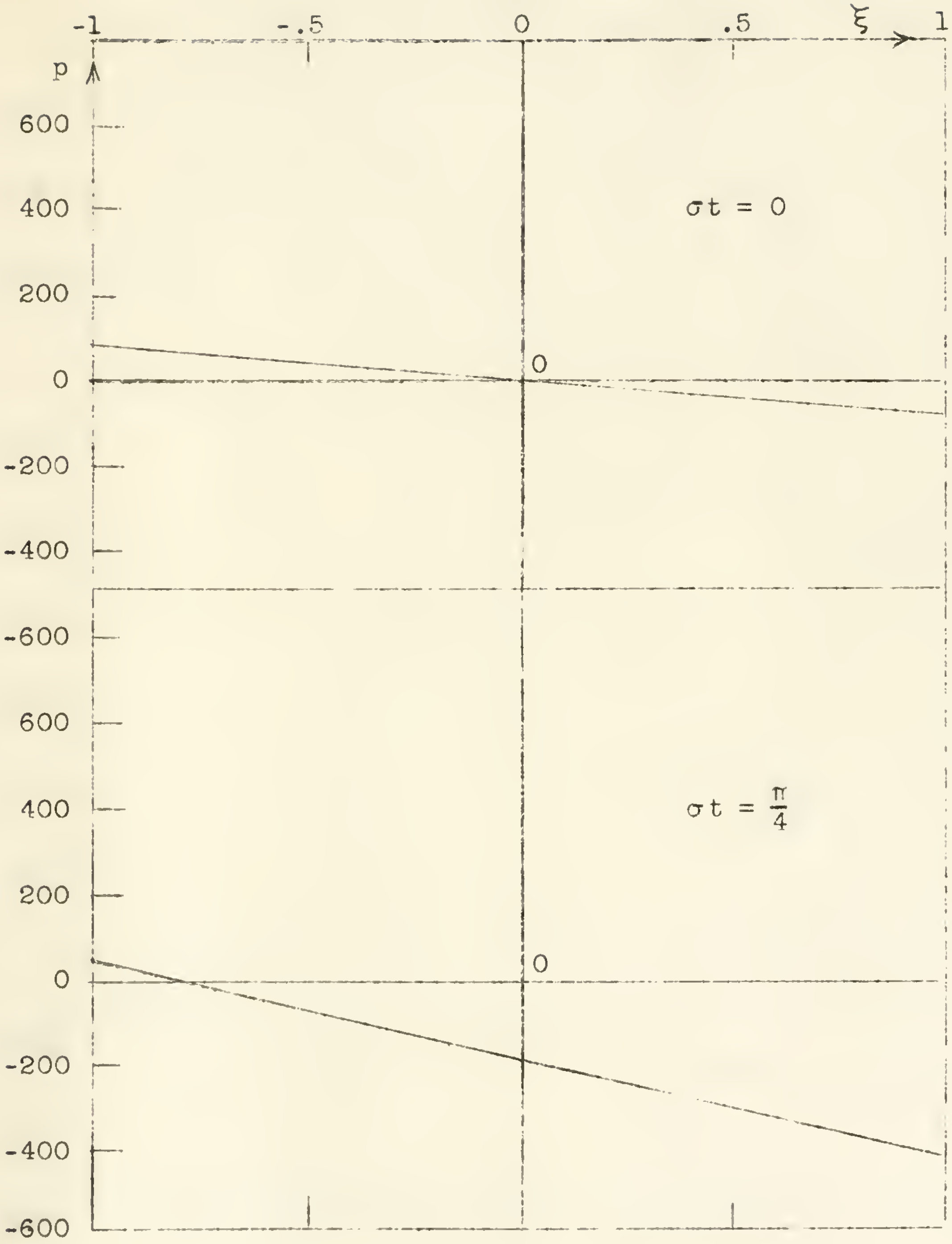


Fig. 2.2

Pressure variations for the stationary board.

$\theta = 1$

p in pounds per foot

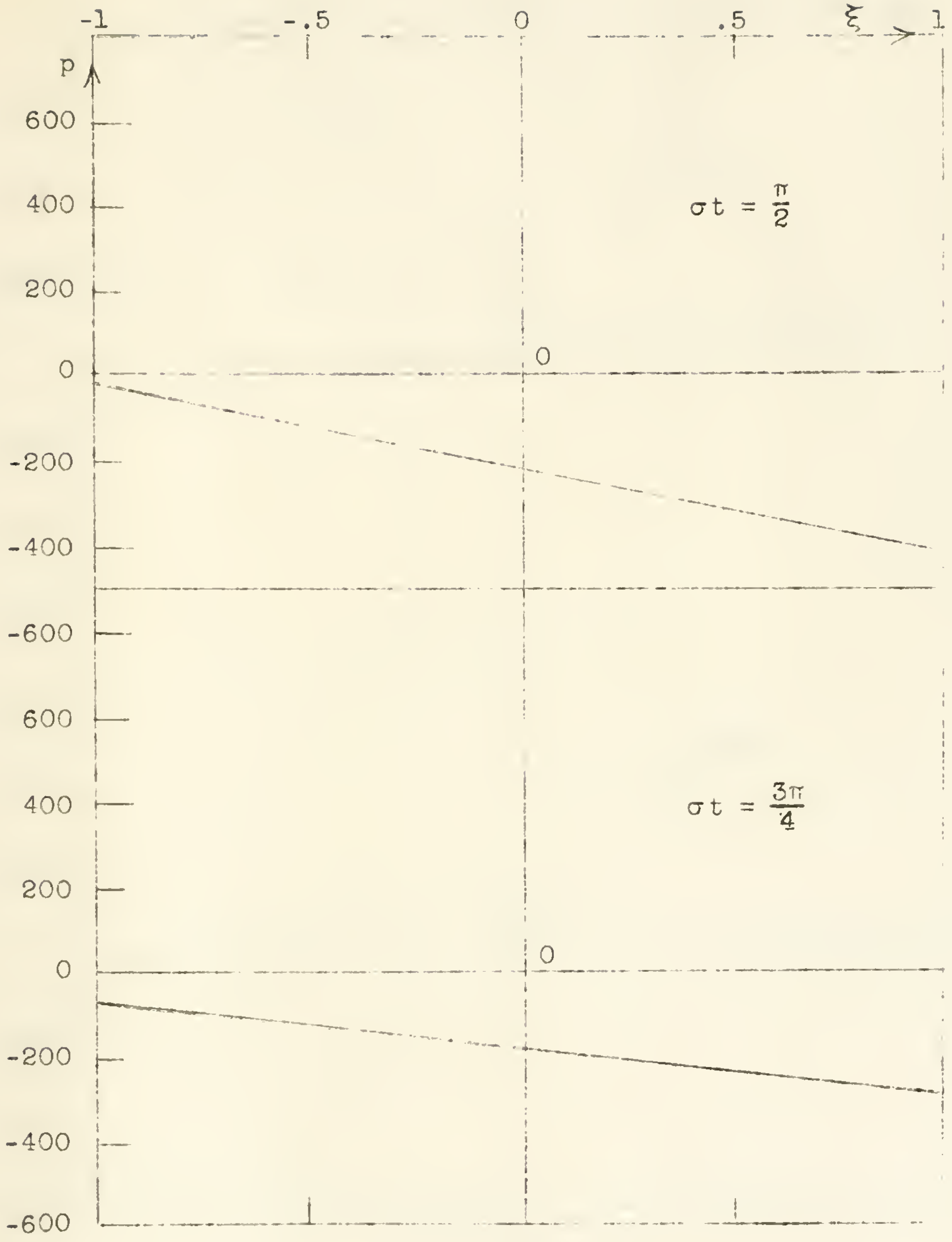

-1
-.5
0
.5
ξ
1
p
600
400
200
0
-200
-400
-600
600
400
200
0
-200
-400
-600
$\sigma t = \frac{\pi}{2}$
0
$\sigma t = \frac{3\pi}{4}$
0

Fig. 2.2 (cont.)

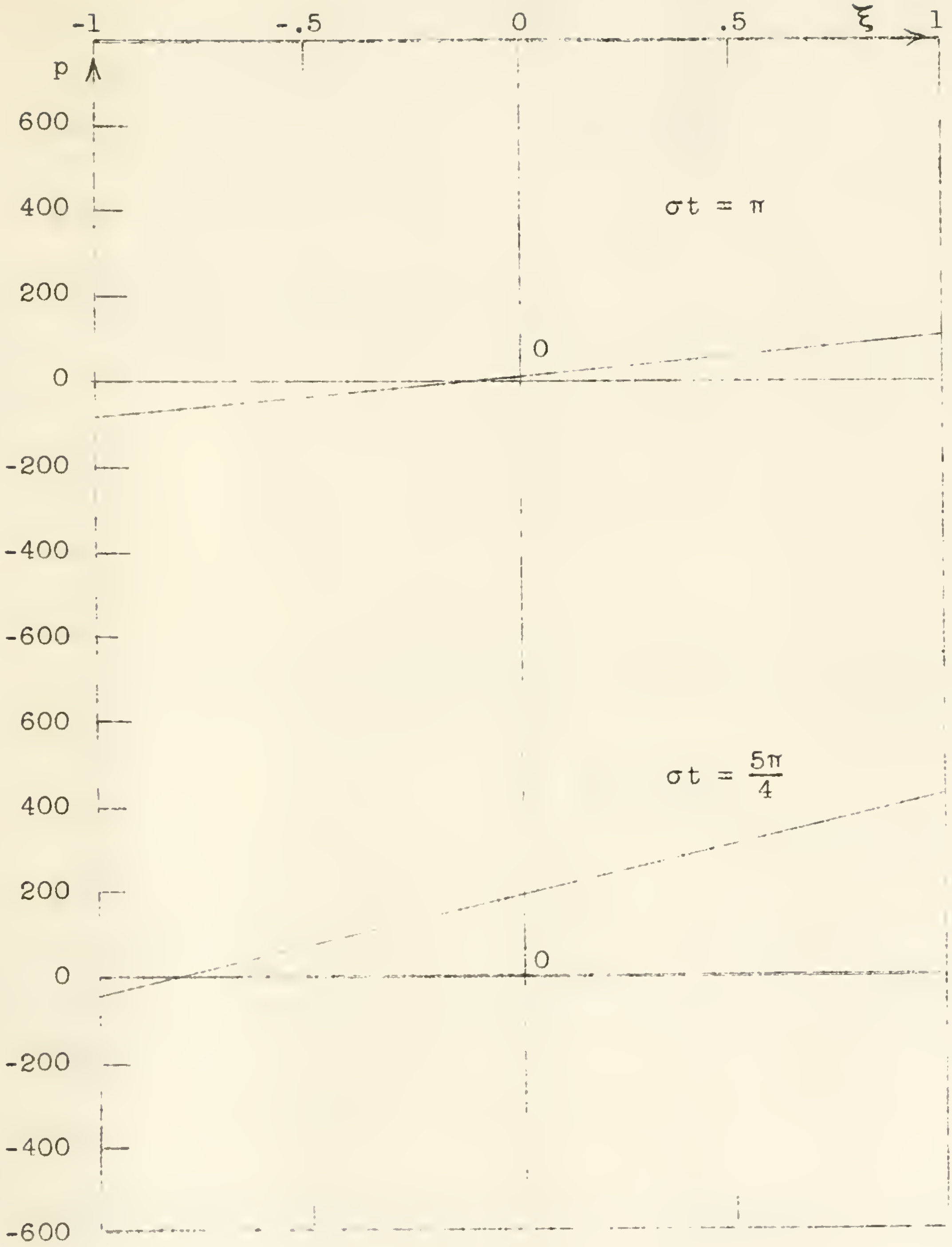

-1
-.5
0
.5
ξ
1
p
600
400
200
0
-200
-400
-600
600
400
200
0
-200
-400
-600
σt = π
0
σt = 5π/4
0

Fig. 2.2 (cont.)

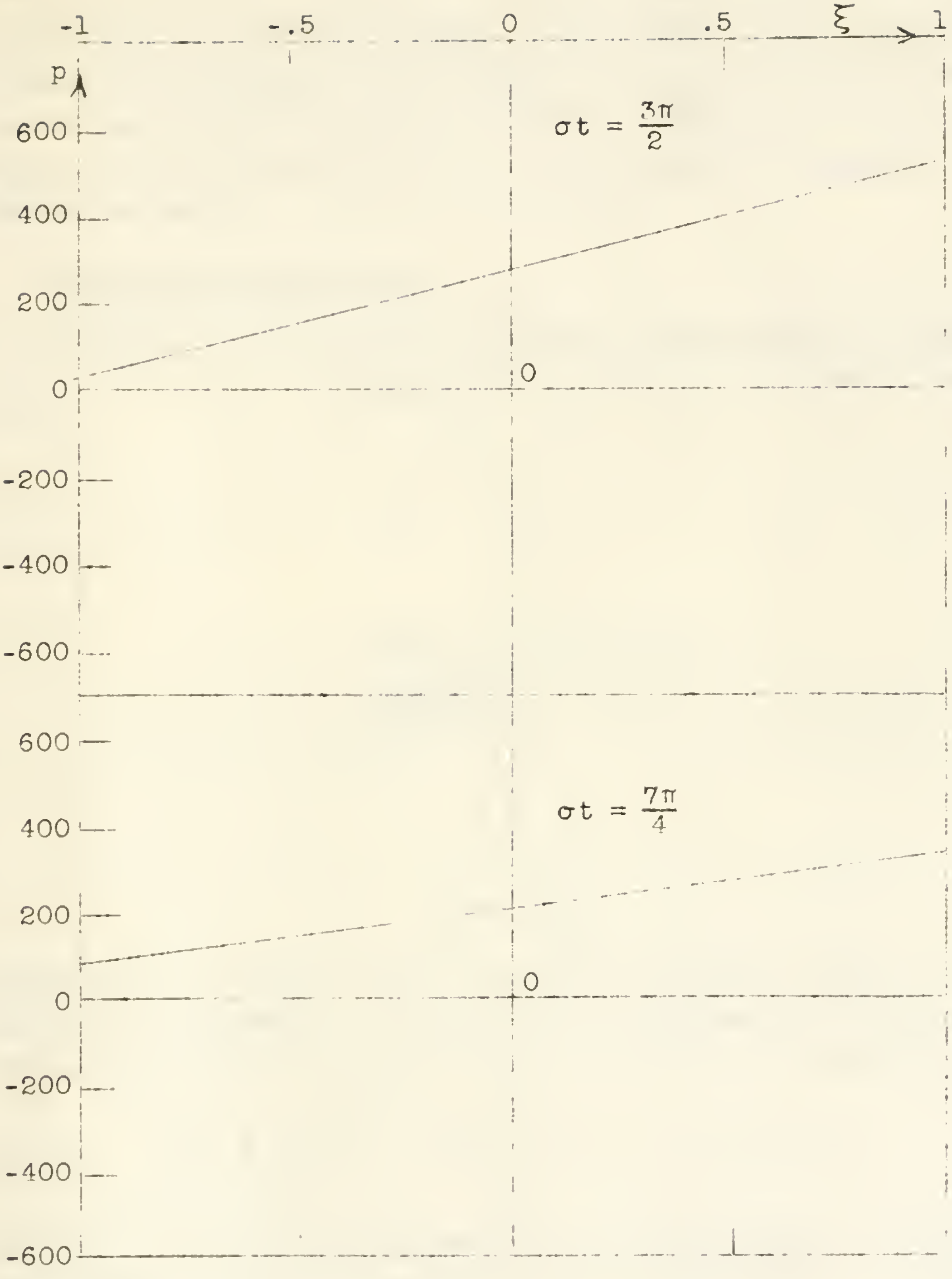

-1
-.5
0
.5
1
ξ
p
600
400
200
0
-200
-400
-600
$\sigma t = \frac{3\pi}{2}$
0
600
400
200
0
-200
-400
-600
$\sigma t = \frac{7\pi}{4}$
0

Fig. 2.2 (cont.)

parameter θ has the value $\theta = 1$, i.e. the length of the board is the same as the wave length. One observes that the pressure variation is greater at the right end than at the left, which is not surprising since the board has a damping effect on the waves. One observes also that the pressure is sometimes less than atmospheric.

b) Freely Floating Board.

In fig. 2.3 the notation for the present case is indicated: $\bar{u}(t)$, $\bar{v}(t)$ represent the coordinates of the center of gravity of the board in the displaced position, and $\omega(t)$ the angular displacement. As before, we consider only simple harmonic oscillations and thus take $\bar{u}$, $\bar{v}$, and ω in the form

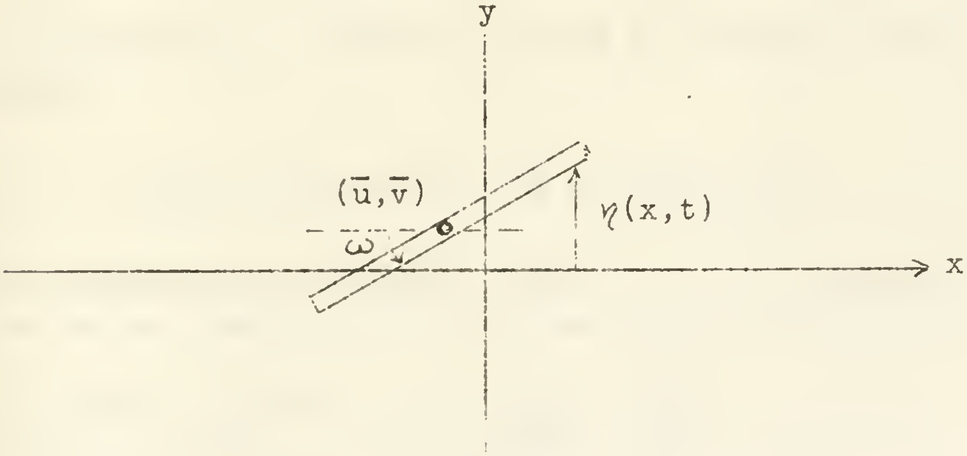


fig. 2.3

$$(2.24) \qquad \bar{u} = \bar{x}\, e^{i\sigma t}, \qquad \bar{v} = \bar{y}\, e^{i\sigma t}, \qquad \omega = \bar{\omega}\, e^{i\sigma t},$$

in which $\bar{x}$, $\bar{y}$, and $\bar{\omega}$ are constants representing the complex amplitudes of these components of the oscillation. For $\eta(x,t)$ we have, therefore

$$(2.25) \qquad \eta(x,t) = [\bar{y} + (x - \bar{x})\bar{\omega}]\, e^{i\sigma t}$$
$$= (\bar{y} + x\bar{\omega})\, e^{i\sigma t}$$

when terms of first order in $\bar{x}$, $\bar{y}$, and $\bar{\omega}$ only are considered. (The horizontal component of the oscillation is thus seen to yield only a second order effect.) The relation (2.25) now yields (cf. 2.7):

$$\phi_{xx} = -\frac{i\sigma}{h}(\bar{y} + x\bar{\omega}) \tag{2.26}$$

in which ϕ is the complex amplitude of the velocity potential $\Phi(x,t) = \phi(x)e^{i\sigma t}$. Hence ϕ is the following cubic polynomial:

$$\phi(x) = -\frac{i\sigma}{h}\left(\frac{\bar{\omega}x^3}{6} + \frac{\bar{y}x^2}{2}\right) + \gamma x + \delta. \tag{2.27}$$

Since the pressure is given by $p = -\rho\,\Phi_t - \rho g\eta$ we have in the present case

$$p(x) = [-i\sigma\,\rho\phi(x) + (\bar{y} + x\bar{\omega})]\, e^{i\sigma t}. \tag{2.28}$$

The transition conditions (2.3) at $x = \pm a$ now lead, in the same way as above from (2.11) and (2.12), to the equations

$$\left\{\begin{aligned} Be^{ika} + Re^{-ika} &= -\frac{i\sigma}{h}\left(\frac{\bar{\omega}a^3}{6} + \frac{\bar{y}a^2}{2}\right) + a\gamma + \delta \\ Be^{ika} - Re^{-ika} &= -\frac{\sigma}{kh}\left(\frac{\bar{\omega}a^2}{2} + \bar{y}a\right) - \frac{i}{k}\gamma \\ Te^{-ika} &= -\frac{i\sigma}{h}\left(-\frac{\bar{\omega}a^3}{6} + \frac{\bar{y}a^2}{2}\right) - a\gamma + \delta \\ Te^{-ika} &= \frac{\sigma}{kh}\left(\frac{\bar{\omega}a^2}{2} - \bar{y}a\right) - \frac{i}{k}\gamma. \end{aligned}\right. \tag{2.29}$$

These four equations are not sufficient to determine the six constants R, T, $\bar{\omega}$, $\bar{y}$, γ, and δ. We must make use of the dynamics of the floating rigid body for this purpose. We have the equations of motion:

$$F = Ma, \quad \text{and} \quad L = I\alpha \tag{2.30}$$

at our disposal. In the first equation F and M are the total

vertical force on the board, and its mass; and a is the vertical acceleration of its center of gravity, I the moment of inertia, and α the angular acceleration. These dynamical conditions then yield the following relations:

$$(2.31) \qquad \int_{-a}^{a} p\,dx = M\ddot{\bar{v}}, \qquad \int_{-a}^{a} px\,dx = I\ddot{\omega},$$

and these in turn lead to the equations

$$(2.32) \quad \left\{ \begin{aligned} &\int_{-a}^{a} \left\{ -i\sigma\rho\left[-\frac{i\sigma}{h}\left(\frac{\bar{\omega}x^3}{6}+\frac{\bar{y}x^2}{2}\right)+\gamma x+\delta\right]+\bar{y}+x\bar{\omega} \right\} dx = -M\sigma^2\bar{y}, \\ &\int_{-a}^{a} \left\{ -i\sigma\rho\left[-\frac{i\sigma}{h}\left(\frac{\bar{\omega}x^4}{6}+\frac{\bar{y}x^3}{2}\right)+\gamma x^2+\delta x\right]+x\bar{y}+x^2\bar{\omega} \right\} dx \\ &\qquad = -I\sigma^2\bar{\theta}. \end{aligned} \right.$$

In the first equation we have ignored the weight of the board, since it is balanced by the hydrostatic pressure. The equations (2.29) and (2.32) now determine all of the unknown complex amplitudes.

We shall give the results for the case of a purely vertical motion without rotation (though the general case presents no special difficulty); i.e. we set $\bar{\omega} = 0$, and solve the above equations. The result is

$$(2.33) \quad \left\{ \begin{aligned} \bar{y} &= \frac{BAD}{E} \\ \delta &= \frac{BAF}{E} \\ \gamma &= \frac{BA}{H} \\ R &= BA^2\left[1+\frac{\theta\pi g}{\sigma}\left(\frac{D}{E}\right)+\frac{ai}{\theta\pi}\left(\frac{1}{H}\right)\right] \\ T &= BA^2\left[\frac{\theta\pi g}{\sigma}\left(\frac{D}{E}\right)-\frac{ai}{\theta\pi}\left(\frac{1}{H}\right)\right] \end{aligned} \right.$$

in which

$$A = e^{\theta\pi i}, \qquad D = -\frac{2i\sigma a}{g},$$

$$E = -\frac{2}{3}\theta^2\pi^2 a + 2a - \frac{\theta^2\pi^2 hM}{a^2\rho} + 2i\theta,$$

$$F = -\frac{\theta^2\pi^2 hM}{a^2\rho} + \frac{\theta^2\pi^2 a}{3} + 2a, \qquad H = \frac{a}{\theta\pi}(\theta\pi - i),$$

$$\theta = \frac{2a}{\lambda}, \qquad \lambda = \frac{2\pi}{k}.$$

From the expressions for R and T in equations (2.19), the reflection and transmission coefficients, C_r and C_t respectively, can be calculated, using $C_r = \frac{|R|}{|B|}$ and $C_t = \frac{|T|}{|B|}$. Again one finds that $C_r^2 + C_t^2 = 1$.

We wish next to study the pressure under the board. For this purpose it is necessary to insert the function ϕ given by (2.27) in (2.28) and take the real part. The calculation is straightforward, but tedious, and gives the result:

$$(2.34) \qquad p(x,t) = p_1(x)\cos\sigma t - p_2(x)\sin\sigma t$$

with

$$(2.35) \qquad \begin{cases} p_1(x) = \dfrac{\sigma\rho B}{C^2 + D^2}[E\sin r + F\cos r] \\ \\ p_2(x) = \dfrac{\sigma\rho B}{C^2 + D^2}[F\sin r - E\cos r]. \end{cases}$$

The quantities C, D, E, F are given by

$$C = 4 + r^2(Q - 1)$$

$$D = -\frac{2}{r} - r(Q - 3)$$

$$\begin{aligned} E = {} & \xi^2[-r^4(Q - 1) - r^2(Q + 1) - 2] \\ (2.36) \qquad & + \xi[r^4(Q - 1)^2 + 4r^2(Q + 4)] \\ & + [r^4Q(Q - 1) + r^2Q(Q + 1) + 2Q] \end{aligned}$$

$$\begin{aligned} F = {} & \xi^2[2r^3 + 2r] + \xi[r^3(Q - 1)^2 + 4rQ + \frac{4}{r}] \\ & + [-2r^3Q - 2rQ] \end{aligned}$$

with

$$r = \theta\pi$$

$$Q = \frac{1}{3} - \frac{hM}{a^3\rho}$$

$$\xi = \frac{x}{a} .$$

In fig. 2.4 the results of calculations for the pressure distribution in a numerical case are given. The parameters were chosen as follows:

$$\theta = 1, \quad h = 1 \text{ ft.}, \quad B = 1, \quad a = 4 \text{ ft.}$$

$$M = 0.3\rho/g, \quad \sigma = 4.46 \text{ rad./sec.}, \quad \lambda = 8 \text{ ft.}$$

A few observations might be made. First of all, we note that in both cases the pressure variation at the right end ($\xi = 1$), where the incoming wave is incident, is greater than at the left end. This is to be expected, since the barrier exercises a damping effect on the wave going under it. The pressure distribution in the case of the floating board is quadratic in x, in contrast with the case of the fixed board

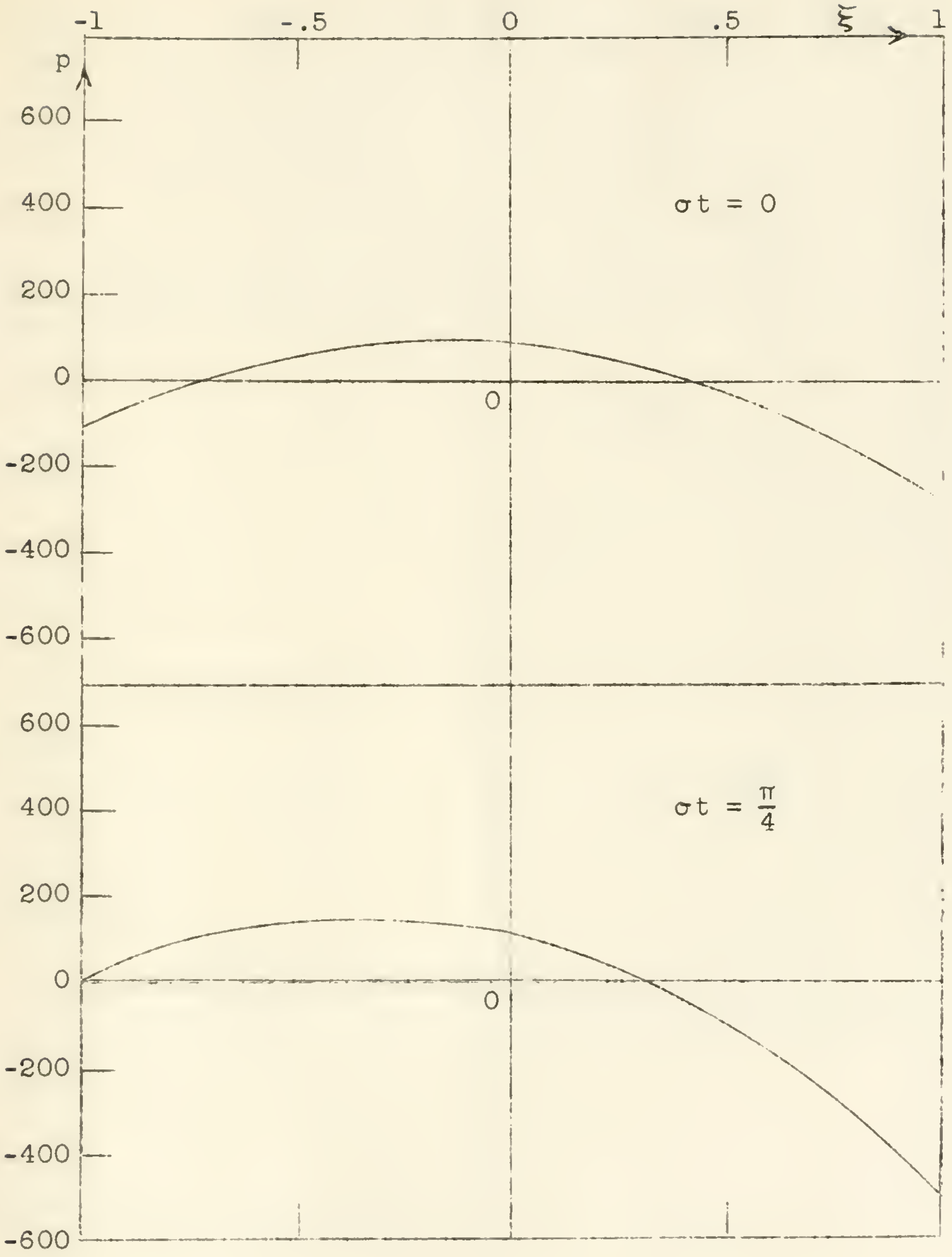


Fig. 2.4

Pressure variations for floating board.

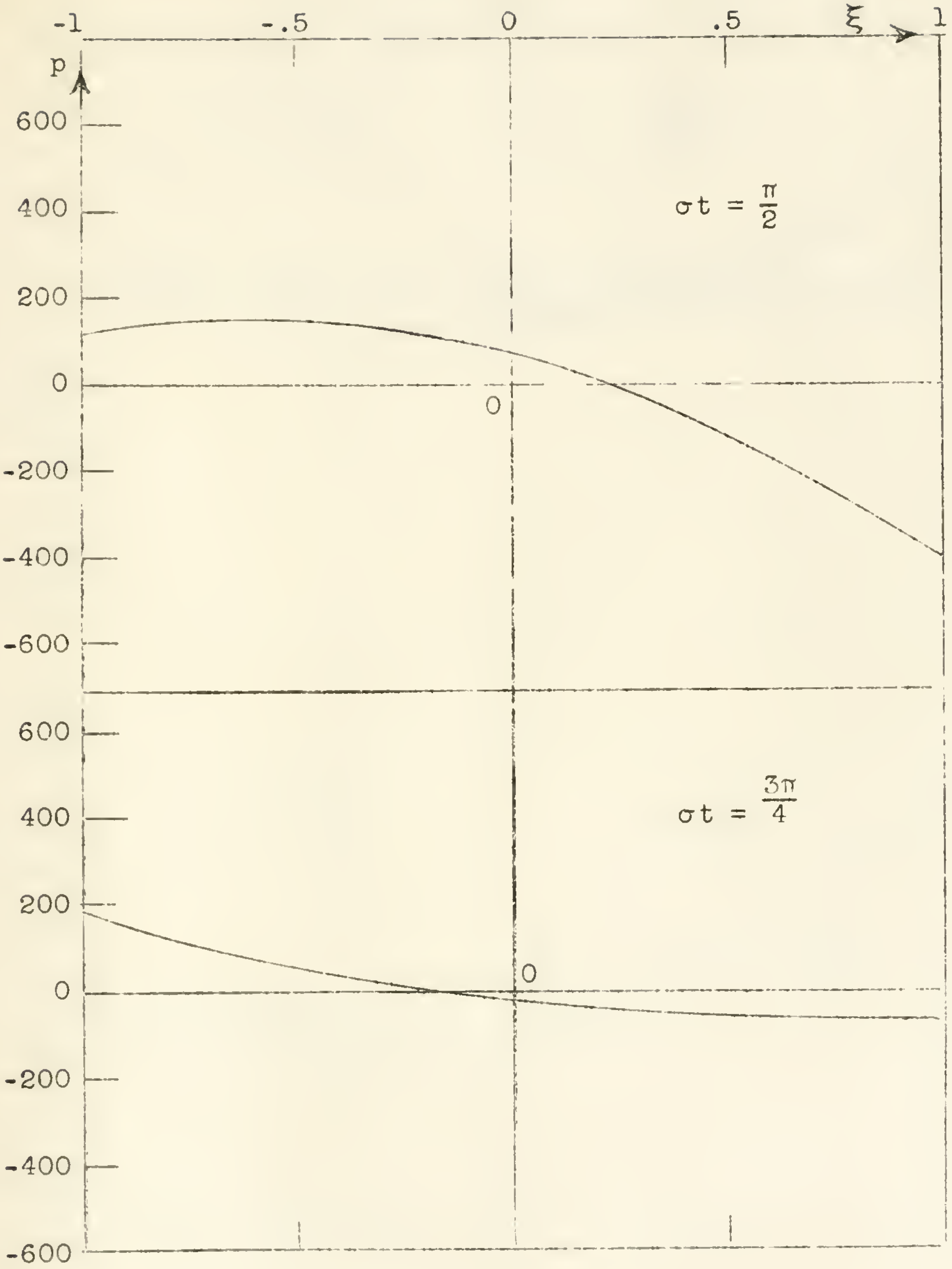
$\theta = 1$

p in pounds per foot

[illegible]

[illegible]

[illegible]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$\sigma t = \frac{\pi}{2}$
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-200
-400
-600
$\sigma t = \frac{3\pi}{4}$
0

Fig. 2.4 (cont.)

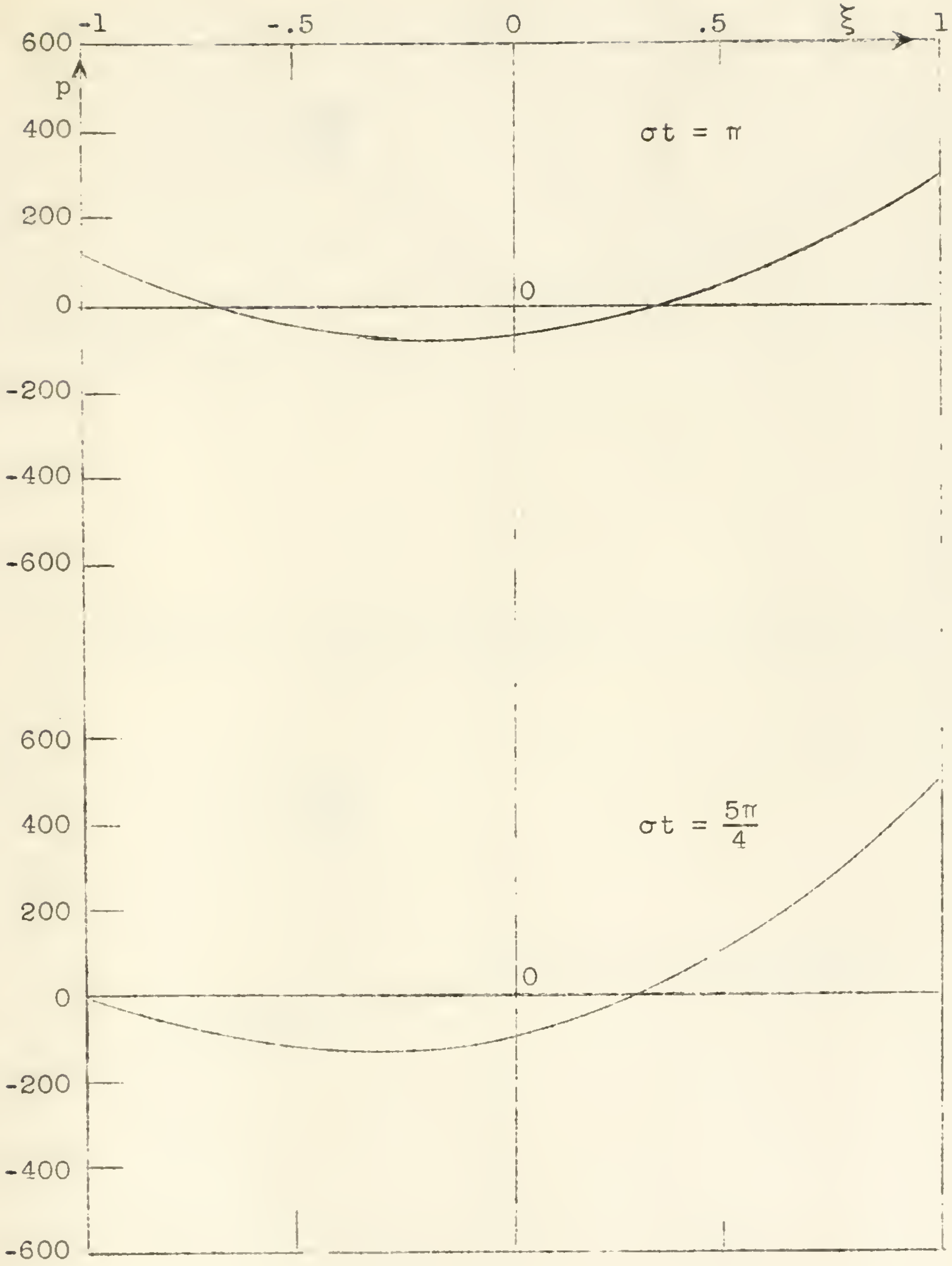

Fig. 2.4 (cont.)

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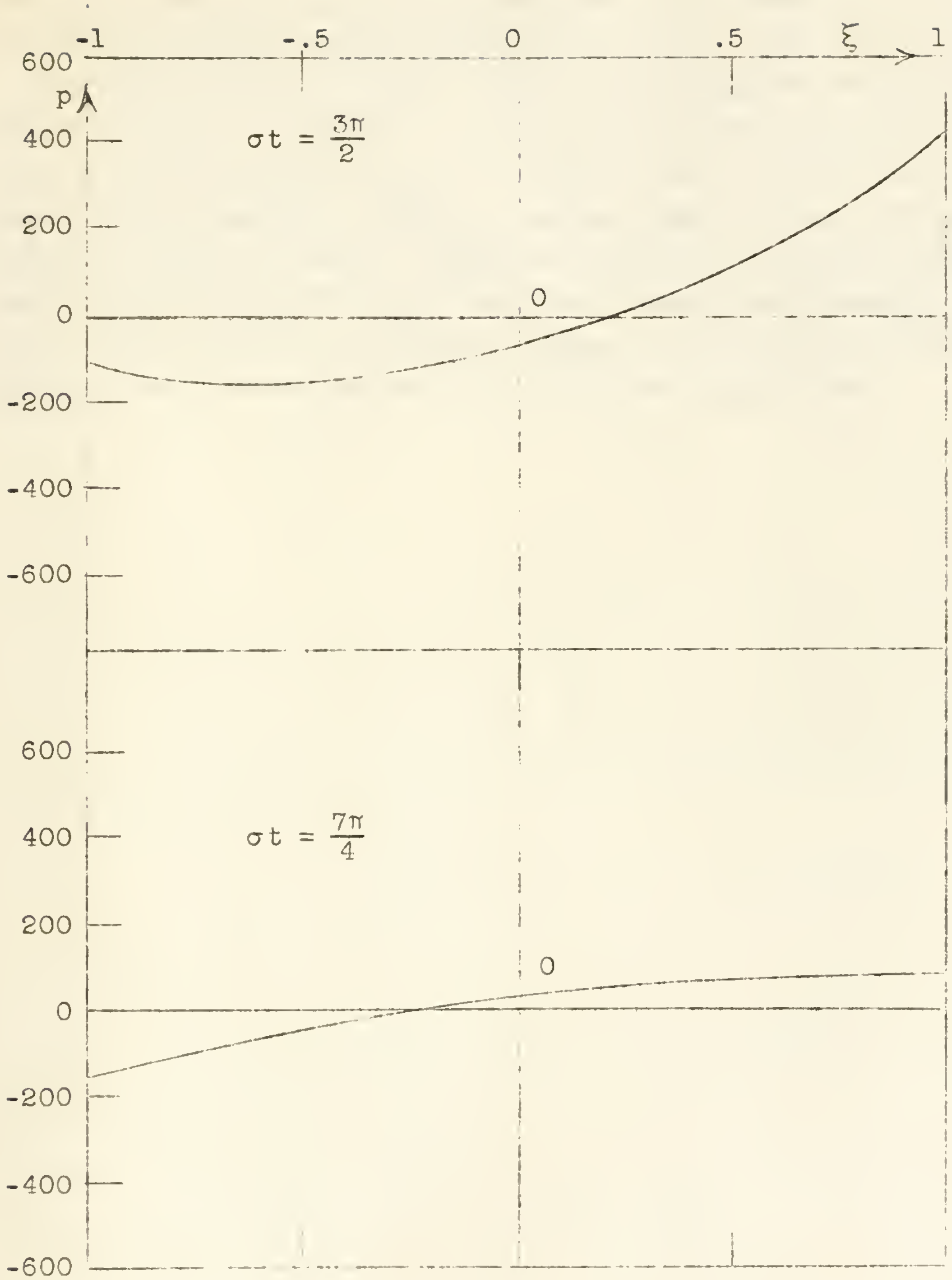
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$\sigma t = \frac{3\pi}{2}$
200
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0
-200
-400
-600
600
400
$\sigma t = \frac{7\pi}{4}$
200
0
0
-200
-400
-600

Fig. 2.4 (cont.)

in which the distribution of pressure was linear in x. Next, we note that the pressure variation at the right end of the stationary board is greater than at the same end of the floating one; this too might be expected since the fixed board receives the full impact of the incident wave, while the floating one yields somewhat. Finally, we see that at the left end the opposite effect occurs: there the pressure variation under the stationary barrier is less than that under the floating one. This is not surprising either, since the fixed board should damp the wave more successfully than the movable one.

[illegible] of pressure [illegible] be [illegible] variation in the [illegible] of the following [illegible] of the [illegible] which [illegible] be expected [illegible] and [illegible] Finally [illegible]

3. The Effect of a Floating Beam with Bending Flexibility on Simple Harmonic Waves.

By L. Weliczker

The beam is assumed to extend from $x = -\ell$ to $x = 0$ and, as in the above section, to be in simple harmonic motion due to an incoming wave from $x = +\infty$. The basic relations for $\Phi(x,t)$ on the free surface, and for $\eta(x,t)$ under the beam are the same as before:

$$(3.1) \qquad \Phi_{xx} = \frac{1}{gh}\Phi_{tt}, \quad x > 0,\ x < -\ell.$$

$$(3.2) \qquad \eta_t = -h\,\Phi_{xx}, \quad -\ell < x < 0.$$

We assume once more that the beam sinks very little below the water surface when in equilibrium (i.e. very little in relation to the depth of the water), so that the coefficient of Φ_{xx} in (3.2) can be taken as h rather than $(h + \bar{\eta})$ (cf. (1.15)), and also the transition conditions at the ends of the beam are

$$(3.3) \qquad \Phi_x \text{ and } \Phi_t \text{ continuous at } x = 0,\ x = -\ell.$$

After writing

$$(3.4) \qquad \Phi(x,t) = \phi(x)e^{i\sigma t}, \quad \eta(x,t) = v(x)e^{i\sigma t}$$

we find, as before:

$$(3.5) \qquad \phi_{xx} + \frac{\sigma^2}{gh}\phi = 0, \quad x > 0,\ x < -\ell$$

$$(3.6) \qquad \phi_{xx} + \frac{i\sigma}{h}v = 0, \quad -\ell < x < 0.$$

The conditions at ∞ have the effect that (cf. (2.10) et seq.):

$$(3.7) \qquad \phi(x) = Be^{ikx} + Re^{-ikx}, \quad x > 0,$$

$$(3.8) \qquad \phi(x) = Te^{ikx}, \quad x < -\ell,$$

with $k = \sigma\sqrt{gh}$. All of this is the same as for the previous cases. We turn now to the conditions which result from the assumption that the floating body is a beam.

The differential equation governing small transverse oscillations of a beam is

$$(3.9) \qquad EI\,\eta_{xxxx} + m\,\eta_{tt} = p,$$

in which E is the modulus of elasticity, I the moment of inertia of a cross section of unit breadth (or, perhaps better, EI is the bending stiffness factor), m the mass per unit area, and p is the pressure. We ignore the weight of the beam and at the same time disregard the contribution of the hydrostatic pressure term in p corresponding to the equilibrium position of the beam---i.e. the pressure here is that due entirely to the dynamics of the situation. Thus

$$(3.10) \qquad \begin{aligned} p &= -\rho\,\Phi_t - pg\eta \\ &= (-i\sigma\rho\phi - pgv)e^{i\sigma t}. \end{aligned}$$

Insertion of this relation in (3.9) and use of (3.6) leads at once to the differential equation for $\phi(x)$:

$$(3.11) \qquad \frac{d^6\phi}{dx^6} + \frac{pg - m\sigma^2}{EI}\frac{d^2\phi}{dx^2} + \frac{\sigma^2\rho}{EIh}\phi = 0, \quad -\ell < x < 0,$$

that is valid under the beam. The case of greatest importance for us---that of a floating beam used as a breakwater---leads obviously to the boundary conditions for the beam which correspond to free ends, i.e. to the conditions that the shear and bending moments should vanish at the ends of the beam. These conditions in turn mean that η_{xx} and η_{xxx}

should vanish at the ends of the beam, and from (3.6) we thus have for ϕ the boundary conditions

$$(3.12) \qquad \frac{d^4\phi}{dx^4} = \frac{d^5\phi}{dx^5} = 0 \quad \text{at} \quad x = 0,\ x = -\ell.$$

The transition conditions (3.3) require that ϕ and ϕ_x be continuous at $x = 0$, $x = -\ell$, and this, in view of (3.7) and (3.8), requires that

$$(3.13) \qquad \begin{cases} \phi(0) = B + R\ , & \phi_x(0) = ikB - ikR \\ \phi(-\ell) = Te^{-ik\ell}, & \phi_x(-\ell) = ikTe^{-ik\ell}. \end{cases}$$

We remark once more that the constant B is assumed real, but that R and T will in general be complex constants, and that the real parts of Φ and η as given by (3.4) are to be taken at the end.

In order to solve our problem we must solve the differential equation (3.11) subject to the conditions (3.12) and (3.13). A count of the relations available to determine the solution should be made: The general solution of (3.11) contains six arbitrary constants, and we wish to determine the constants R and T (the amplitudes of the reflected and transmitted waves) occurring in (3.13) once the constant B (the amplitude of the incoming wave) has been fixed. In all there are thus eight constants to be found, and we have in (3.12) and (3.13) eight relations to determine them. Once these constants have been determined, the reflection and transmission coefficients are known, and the deflection of the beam can be found from (3.6). The maximum bending stresses in the beam can then be calculated from the usual formula, $s = Mc/I$, with $M = EI\,\eta_{xx}$ and c the distance from the neutral axis to the extreme outer fibres of the beam.

In principle, therefore, the solution of the problem is straightforward. However, the carrying out of the details in the case of the beam of finite length is very tedious, involving as it does a system of eight linear equations for eight unknowns with complex coefficients. In addition, one must determine the roots of a sixth degree algebraic equation in order to find the general solution of (3.11). These roots are in general complex numbers and they involve the essential parameters of the mechanical system. Thus it is clear that a discussion of the behavior of the system in general terms with respect to arbitrary values of the parameters of the system is not feasible, and one must turn rather to concrete cases in which most of the parameters have been given specific numerical values. The results of some calculations of this kind have already been given in the summary at the beginning of this paper, though only for the reflection coefficient. The determination of the deflection of the beam, though straightforward, would involve a great deal more labor than is required to determine the reflection coefficient.

The case of a semi-infinite beam---i.e. a beam extending from $x = 0$ to $x = -\infty$---is simpler to deal with in that the conditions in the second line of (3.13) fall away, and the conditions (3.12) at $x = -\ell$ can be replaced by the requirement that ϕ be bounded at $x = -\infty$. The number of constants to be fixed then reduces to four instead of eight, but the determination of the deflection of the beam still remains a formidable problem; we shall consider this case as well as the case of a beam of finite length.

We begin the program indicated with a discussion of the general solution of the differential equation (3.11). Since it is a linear differential equation with constant coefficients we proceed in the standard fashion by setting $\phi = e^{\kappa x}$, inserting in (3.11), to find for κ the equation

$$(3.14) \qquad \kappa^6 + a\kappa^2 + b = 0$$

with

$$(3.15) \qquad a = \frac{\rho g - m\sigma^2}{EI} \quad , \quad b = \frac{\sigma^2 \rho}{EIh} \; .$$

This is a cubic equation in $\kappa^2 = \beta$, which happens to be in the standard form to which the Cardan formula for the roots of a cubic applies directly. For the roots β_i of this cubic equation one has therefore

$$(3.16) \qquad \begin{aligned} \beta_1 &= u + v \\ \beta_2 &= \varepsilon u + \varepsilon^2 v \\ \beta_3 &= \varepsilon^2 u + \varepsilon v \end{aligned}$$

with u and v defined by

$$(3.17) \qquad u = \left| -\frac{b}{2} + \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}} \right|^{\frac{1}{3}} , \quad v = \left| -\frac{b}{2} - \left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{\frac{1}{2}} \right|^{\frac{1}{3}}$$

and ε the cube root of unity:

$$(3.18) \qquad \varepsilon = \frac{-1 + i\sqrt{3}}{2} \; .$$

The constant a is positive, since σ, the frequency of the incoming wave, is so small in the cases of interest in practice that ρg is much larger than $m\sigma^2$. The constant b is obviously positive. Consequently the root β_1 is real and negative since $|u| < |v|$ and v is negative. Thus the roots $\kappa_1 = + \beta_1^{1/2}$, $\kappa_2 = -\beta_1^{1/2}$ are pure imaginary. The quantities β_2 and β_3 are complex conjugates, and their square roots yield two pairs of complex conjugates

$\kappa_3 = + \beta_2^{1/2}$, $\kappa_4 = - \beta_2^{1/2}$, $\kappa_5 = + \beta_3^{1/2}$, $\kappa_6 = - \beta_3^{1/2}$.

For β_2 and β_3 we have

$$(3.19) \qquad \beta_2 = -\frac{1}{2}(u + v) + i\frac{\sqrt{3}}{2}(u - v),$$

$$(3.20) \qquad \beta_3 = -\frac{1}{2}(u + v) - i\frac{\sqrt{3}}{2}(u - v).$$

Thus β_2 and β_3 both have positive real parts. We suppose the roots κ_3, κ_4, κ_5, κ_6 to be numbered so that κ_3 and κ_5 are taken to have positive real parts, while κ_4 and κ_6 have negative real parts. The general solution of (3.11) thus is

$$(3.21) \qquad \phi(x) = a_1e^{\kappa_1 x} + a_2e^{\kappa_2 x} + a_3e^{\kappa_3 x} + a_4e^{\kappa_4 x} + a_5e^{\kappa_5 x} + a_6e^{\kappa_6 x}.$$

In the case of a beam covering the whole surface of the water, i.e. extending from $-\infty$ to $+\infty$, the condition that ϕ be bounded at $x = \pm\infty$ would require that $a_3 = a_4 = a_5 = a_6 = 0$ since the exponentials in the corresponding terms have non-vanishing real parts. The remaining terms yield progressing waves traveling in opposite directions; their wave lengths are given by $\lambda = 2\pi/|\kappa_1| = 2\pi/|\kappa_2|$ and thus by

$$(3.22) \qquad \lambda = 2\pi/\sqrt{|u + v|},$$

with $u + v$ defined by (3.17). The wave length and frequency are thus connected by a rather complicated relation, and, unlike the case of waves in shallow water with no immersed bodies or constraints on the free surface, the wave length is not independent of the frequency and the wave phenomena are subject to dispersion.

In the case of a beam extending from the origin to $-\infty$ while the water surface is free for $x > 0$, the boundedness condition for ϕ at $-\infty$ requires that we take $a_4 = a_6 = 0$ since κ_4 and κ_6 have negative real parts and consequently $e^{\kappa_4 x}$ and $e^{\kappa_6 x}$ would yield exponentially unbounded contributions to ϕ at $x = -\infty$. We know that κ_1 and κ_2 are pure imaginary with opposite signs, with κ_2, say, negative imaginary. Since no progressing wave is assumed to come from the left, we must then take $a_2 = 0$. Thus the term $a_1 e^{\kappa_1 x}$ yields the transmitted wave and the terms involving a_3 and a_5 yield disturbances which die out exponentially at ∞. The conditions (3.12) and (3.13) at $x = 0$ now yield the following four linear equations:

$$
\begin{aligned}
\kappa_1^4 a_1 + \kappa_3^4 a_3 + \kappa_5^4 a_5 &= 0 \\
\kappa_1^5 a_1 + \kappa_3^5 a_3 + \kappa_5^5 a_5 &= 0 \\
a_1 + a_3 + a_5 &= B + R \\
\kappa_1 a_1 + \kappa_3 a_3 + \kappa_5 a_5 &= ik(B - R)
\end{aligned}
\tag{3.22}
$$

for the constants a_1, a_3, a_5, R. For the amplitude R of the reflected wave one finds

$$
\frac{R}{B} = \frac{\begin{vmatrix} -1 & -1 & -1 & -1 \\ \kappa_1 & \kappa_3 & \kappa_5 & ik \\ \kappa_1^4 & \kappa_3^4 & \kappa_5^4 & 0 \\ \kappa_1^5 & \kappa_3^5 & \kappa_5^5 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & -1 & -1 & -1 \\ \kappa_1 & \kappa_3 & \kappa_5 & ik \\ \kappa_1^4 & \kappa_3^4 & \kappa_5^4 & 0 \\ \kappa_1^5 & \kappa_3^5 & \kappa_5^5 & 0 \end{vmatrix}}
\tag{3.23}
$$

Even in this relatively simple case of the semi-infinite beam the reflection coefficient is so complicated a function of the parameters (even though it is algebraic in them) that it seems not worthwhile to write it down explicitly. The results of numerical calculations based on (3.23) have already been given in Tables A, B, C, and D of the summary at the beginning of this paper.

In the case of the beam of finite length extending from $x = -\ell$ to $x = 0$ the eight conditions given by (3.12) and (3.13) must be satisfied by the solution (3.21) of the differential equation (3.11), and these conditions serve to determine the six constants of integration and the amplitudes R and T of the reflected and transmitted waves. The problem thus posed is quite straightforward but extremely tedious as it involves solving eight linear equations for eight complex constants. The amplitude R of the reflected wave, for example, is given by the ratio of the following eighth order determinants:

$$\frac{R}{B} = \frac{\begin{vmatrix} \kappa_1^4 & \kappa_1^4 & \kappa_3^4 & \kappa_3^4 & \kappa_5^4 & \kappa_5^4 & 0 & 0 \\ \kappa_1^5 & -\kappa_1^5 & \kappa_3^5 & -\kappa_3^5 & \kappa_5^5 & -\kappa_5^5 & 0 & 0 \\ \kappa_1^4 e^{-\kappa_1 \ell} & \kappa_1^4 e^{\kappa_1 \ell} & \kappa_3^4 e^{-\kappa_3 \ell} & \kappa_3^5 e^{\kappa_3 \ell} & \kappa_5^4 e^{-\kappa_5 \ell} & \kappa_5^4 e^{\kappa_5 \ell} & 0 & 0 \\ \kappa_1^5 e^{-\kappa_1 \ell} & \kappa_1^5 e^{\kappa_1 \ell} & \kappa_3^5 e^{-\kappa_3 \ell} & \kappa_3^5 e^{\kappa_3 \ell} & \kappa_5^5 e^{-\kappa_5 \ell} & \kappa_5^5 e^{\kappa_5 \ell} & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ \kappa_1 & -\kappa_1 & \kappa_3 & -\kappa_3 & \kappa_5 & -\kappa_5 & ik & 0 \\ e^{-\kappa_1 \ell} & e^{\kappa_1 \ell} & e^{-\kappa_3 \ell} & e^{\kappa_3 \ell} & e^{-\kappa_5 \ell} & e^{\kappa_5 \ell} & 0 & -e^{-ik\ell} \\ \kappa_1 e^{-\kappa_1 \ell} & -\kappa_1 e^{\kappa_1 \ell} & \kappa_3 e^{-\kappa_3 \ell} & -\kappa_3 e^{\kappa_3 \ell} & \kappa_5 e^{-\kappa_5 \ell} & -\kappa_5 e^{\kappa_5 \ell} & 0 & -ike^{-ik\ell} \end{vmatrix}}{\Delta},$$

in which $\triangle$ differs from the determinant in the numerator only in the term in the seventh column and fifth row which is $+1$ rather than -1. This formula was used to calculate the values given in the introduction and summary at the beginning of this report.

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